Exercise 18

Obtain the solutions for the velocity potential $\phi(x,z,t)$ and the free surface elevation $\eta(x,t)$ involved in the two-dimensional surface waves in water of finite (or infinite) depth h. The governing equation, boundary, and free surface conditions and initial conditions (see Debnath 1994, p. 92) are

$$\phi_{xx} + \phi_{zz} = 0, \quad -h \le z \le 0,\ -\infty < x < \infty,\ t > 0,$$

$$\left.\begin{aligned} &\phi_t + g\eta = -\frac{P}{\rho}p(x)\exp(i\omega t), \\ &\phi_z - \eta_t = 0 \end{aligned}\right\} \quad z = 0,\ t > 0,$$

$$\phi(x,z,0) = 0 = \eta(x,0) \quad \text{for all } x \text{ and } z.$$

Solution

Depending whether the water has finite or infinite depth, the boundary condition will be different for each case.

$$\text{Boundary condition for finite depth } h: \quad \left.\frac{\partial \phi}{\partial z}\right|_{z=-h} = 0 \tag{1}$$

$$\text{Boundary condition for infinite depth}: \quad \lim_{z\to-\infty} \frac{\partial \phi}{\partial z} = 0 \tag{2}$$

Water of Finite Depth h

The PDEs for ϕ and η are defined for $-\infty < x < \infty$, so we can apply the Fourier transform to solve them. We define the Fourier transform with respect to x here as

$$\mathcal{F}_x\{\phi(x,z,t)\} = \Phi(k,z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\phi(x,z,t)\,dx,$$

which means the partial derivatives of ϕ with respect to x, z, and t transform as follows.

$$\mathcal{F}_x\left\{\frac{\partial^n \phi}{\partial x^n}\right\} = (ik)^n\Phi(k,z,t)$$

$$\mathcal{F}_x\left\{\frac{\partial^n \phi}{\partial z^n}\right\} = \frac{d^n\Phi}{dz^n}$$

$$\mathcal{F}_x\left\{\frac{\partial^n \phi}{\partial t^n}\right\} = \frac{d^n\Phi}{dt^n}$$

Take the Fourier transform of both sides of the first PDE.

$$\mathcal{F}_x\{\phi_{xx} + \phi_{zz}\} = \mathcal{F}\{0\}$$

The Fourier transform is a linear operator.

$$\mathcal{F}_x\{\phi_{xx}\} + \mathcal{F}_x\{\phi_{zz}\} = 0$$

Transform the derivatives with the relations above.

$$(ik)^2\Phi + \frac{d^2\Phi}{dz^2} = 0$$

Expand the coefficient of Φ.

$$-k^2\Phi + \frac{d^2\Phi}{dz^2} = 0$$

Bring the term with Φ to the right side.

$$\frac{d^2\Phi}{dz^2} = k^2\Phi$$

We can write the solution to this ODE in terms of exponentials.

$$\Phi(k,z,t) = A(k,t)e^{|k|z} + B(k,t)e^{-|k|z}$$

We can use boundary condition (1) here to figure out one of the constants. First take the Fourier transform of both sides of it.

$$\mathcal{F}_x\left\{\left.\frac{\partial\phi}{\partial z}\right|_{z=-h}\right\} = \mathcal{F}_x\{0\}$$

Transform the partial derivative.

$$\left.\frac{d\Phi}{dz}\right|_{z=-h} = 0$$

Differentiating Φ with respect to z, we obtain

$$\frac{d\Phi}{dz}(k,z,t) = A(k,t)|k|e^{|k|z} - B(k,t)|k|e^{-|k|z},$$

so the boundary condition gives

$$A(k,t)|k|e^{-|k|h} - B(k,t)|k|e^{|k|h} = 0 \quad \rightarrow \quad A(k,t) = B(k,t)e^{2h|k|},$$

which means

$$\Phi(k,z,t) = B(k,t)e^{-|k|z}(1 + e^{2(h+z)|k|}). \tag{3}$$

Take the Fourier transform with respect to x of the boundary conditions at $z = 0$ now.

$$\mathcal{F}_x\{\phi_t + g\eta\} = \mathcal{F}_x\left\{-\frac{P}{\rho}p(x)e^{i\omega t}\right\}$$
$$\mathcal{F}_x\{\phi_z - \eta_t\} = \mathcal{F}_x\{0\}$$

Use the linearity property.

$$\mathcal{F}_x\{\phi_t\} + g\mathcal{F}_x\{\eta\} = -\frac{P}{\rho}\tilde{p}(k)e^{i\omega t}$$
$$\mathcal{F}_x\{\phi_z\} - \mathcal{F}_x\{\eta_t\} = 0$$

Transform the partial derivatives.

$$\frac{d\Phi}{dt} + gH = -\frac{P}{\rho}\tilde{p}(k)e^{i\omega t} \tag{4}$$
$$\frac{d\Phi}{dz} - \frac{dH}{dt} = 0$$

Differentiate both sides of the first equation with respect to t and move dH/dt to the right side.

$$\frac{d^2\Phi}{dt^2} + g\frac{dH}{dt} = -\frac{P}{\rho}\tilde{p}(k)i\omega e^{i\omega t}$$
$$\frac{d\Phi}{dz} = \frac{dH}{dt}$$

Substitute the second equation into the first.

$$\frac{d^2\Phi}{dt^2} + g\frac{d\Phi}{dz} = -\frac{P}{\rho}\tilde{p}(k)i\omega e^{i\omega t} \tag{5}$$

Evaluate the derivatives of $\Phi(k,z,t)$ in equation (3).

$$\frac{d^2\Phi}{dt^2} = B_{tt}e^{-|k|z}(1+e^{2(h+z)|k|}) \quad \rightarrow \quad \left.\frac{d^2\Phi}{dt^2}\right|_{z=0} = B_{tt}(1+e^{2h|k|})$$
$$\frac{d\Phi}{dz} = B|k|e^{-|k|z}(-1+e^{2(h+z)|k|}) \quad \rightarrow \quad \left.\frac{d\Phi}{dz}\right|_{z=0} = B|k|(-1+e^{2h|k|})$$

Plug these expressions for the derivatives into equation (5).

$$B_{tt}(1+e^{2h|k|}) + gB|k|(-1+e^{2h|k|}) = -\frac{P}{\rho}\tilde{p}(k)i\omega e^{i\omega t}$$

Divide both sides by $1+e^{2h|k|}$.

$$\frac{d^2B}{dt^2} + g|k|\frac{e^{2h|k|}-1}{e^{2h|k|}+1}B = -\frac{i\omega P\tilde{p}(k)}{\rho(e^{2h|k|}+1)}e^{i\omega t}$$

This is a second-order inhomogeneous ODE with constant coefficients. As such, the general solution is obtained by adding the complementary and particular solutions.

$$B(k,t) = B_c + B_p$$

B_c is the solution to the associated homogeneous ODE,

$$\frac{d^2B_c}{dt^2} + g|k|\frac{e^{2h|k|}-1}{e^{2h|k|}+1}B_c = 0,$$

and B_p is the solution that satisfies the inhomogeneous ODE. The function of t on the right side is $e^{i\omega t}$, so B_p has the form $Fe^{i\omega t}$. F is a constant that we determine by plugging this form into the ODE. In the end we get

$$B(k,t) = C_1(k)e^{t\frac{\sqrt{g|k|-e^{2h|k|}g|k|}}{\sqrt{1+e^{2h|k|}}}} + C_2(k)e^{-t\frac{\sqrt{g|k|-e^{2h|k|}g|k|}}{\sqrt{1+e^{2h|k|}}}} + \frac{i\omega P\tilde{p}(k)}{\rho[\omega^2+g|k|+e^{2h|k|}(\omega^2-g|k|)]}e^{i\omega t}$$

for the solution. Resonance occurs if

$$\omega^2 = g|k|\frac{e^{2h|k|}-1}{e^{2h|k|}+1} = g|k|\tanh h|k|$$

in which case the solution is unbounded. The next order of business is to determine C_1 and C_2 using the initial conditions, $\phi(x,z,0)=0$ and $\eta(x,0)=0$. Take the Fourier transform of both sides of these conditions.

$$\begin{aligned}\mathcal{F}_x\{\phi(x,z,0)\} &= \mathcal{F}_x\{0\} \quad \rightarrow \quad \Phi(k,z,0)=0 \\ \mathcal{F}_x\{\eta(x,0)\} &= \mathcal{F}_x\{0\} \quad \rightarrow \quad H(k,0)=0\end{aligned}$$

Using the first one, we get

$$\Phi(k,z,0) = B(k,0)e^{-|k|z}(1+e^{2(h+z)|k|}) = 0,$$

which means

$$B(k,0) = C_1(k) + C_2(k) + \frac{i\omega P\tilde{p}(k)}{\rho[\omega^2 + g|k| + e^{2h|k|}(\omega^2 - g|k|)]} = 0,$$

so

$$C_1(k) = -C_2(k) - \frac{i\omega P\tilde{p}(k)}{\rho[\omega^2 + g|k| + e^{2h|k|}(\omega^2 - g|k|)]}.$$

Solve equation (4) for H now.

$$H(k,t) = \frac{1}{g}\left[-\frac{P}{\rho}\tilde{p}(k)e^{i\omega t} - \frac{d\Phi}{dt}\right]\bigg|_{z=0}$$

Using the second condition, $H(k,0)=0$, we find the second constant $C_2(k)$.

$$C_2(k) = -\frac{iP\tilde{p}(k)}{2\rho[\omega(1+e^{2h|k|}) - i\sqrt{1+e^{2h|k|}}\sqrt{1-e^{2h|k|}}\sqrt{g|k|}]}$$

Putting all this together and simplifying, we therefore have

$$\begin{aligned}\Phi(k,z,t) &= \frac{ie^{-z|k|}}{\rho}P\tilde{p}(k)[1+e^{2(h+z)|k|}]\times \\ &\quad \left[\omega\frac{e^{i\omega t} - e^{it\sqrt{g|k|\tanh h|k|}}}{\omega^2 + g|k| + e^{2h|k|}(\omega^2 - g|k|)} + \frac{i\sin(t\sqrt{g|k|\tanh h|k|})}{\omega(1+e^{2h|k|}) - i\sqrt{1+e^{2h|k|}}\sqrt{1-e^{2h|k|}}\sqrt{g|k|}}\right] \\ H(k,t) &= \sqrt{\frac{|k|}{g}}\frac{i\sqrt{1+e^{2h|k|}}\sqrt{1-e^{2h|k|}}P\tilde{p}(k)}{(1+e^{2h|k|})\rho\omega^2 - (-1+e^{2h|k|})g\rho|k|}\alpha(k,t) + \frac{P|k|\tilde{p}(k)}{\rho\omega^2\coth h|k| - g\rho|k|}e^{i\omega t},\end{aligned}$$

where

$$\begin{aligned}\alpha(k,t) &= \omega e^{it\sqrt{g|k|\tanh h|k|}} - \frac{[\omega^2 + g|k| + e^{2h|k|}(\omega^2 - g|k|)]\cos(t\sqrt{g|k|\tanh h|k|})}{\omega(1+e^{2h|k|}) - i\sqrt{1+e^{2h|k|}}\sqrt{1-e^{2h|k|}}\sqrt{g|k|}} \\ \tilde{p}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} p(x)e^{-ikx}\,dx.\end{aligned}$$

To get $\phi(x,z,t)$ and $\eta(x,t)$, take the inverse Fourier transform of Φ and H, respectively.

$$\phi(x,z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(k,z,t)e^{ikx}\,dk \quad \text{and} \quad \eta(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}H(k,t)e^{ikx}\,dk$$

Water of Infinite Depth

The PDEs for ϕ and η are defined for $-\infty < x < \infty$, so we can apply the Fourier transform to solve them. We define the Fourier transform with respect to x here as

$$\mathcal{F}_x\{\phi(x,z,t)\} = \Phi(k,z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\phi(x,z,t)\,dx,$$

which means the partial derivatives of ϕ with respect to x, z, and t transform as follows.

$$\mathcal{F}_x\left\{\frac{\partial^n\phi}{\partial x^n}\right\} = (ik)^n\Phi(k,z,t)$$
$$\mathcal{F}_x\left\{\frac{\partial^n\phi}{\partial z^n}\right\} = \frac{d^n\Phi}{dz^n}$$
$$\mathcal{F}_x\left\{\frac{\partial^n\phi}{\partial t^n}\right\} = \frac{d^n\Phi}{dt^n}$$

Take the Fourier transform of both sides of the first PDE.

$$\mathcal{F}_x\{\phi_{xx}+\phi_{zz}\} = \mathcal{F}\{0\}$$

The Fourier transform is a linear operator.

$$\mathcal{F}_x\{\phi_{xx}\} + \mathcal{F}_x\{\phi_{zz}\} = 0$$

Transform the derivatives with the relations above.

$$(ik)^2\Phi + \frac{d^2\Phi}{dz^2} = 0$$

Expand the coefficient of Φ.

$$-k^2\Phi + \frac{d^2\Phi}{dz^2} = 0$$

Bring the term with Φ to the right side.

$$\frac{d^2\Phi}{dz^2} = k^2\Phi$$

We can write the solution to this ODE in terms of exponentials.

$$\Phi(k,z,t) = A(k,t)e^{|k|z} + B(k,t)e^{-|k|z}$$

We can use boundary condition (2) here to figure out one of the constants. Taking the Fourier transform with respect to x of both sides of it gives us

$$\mathcal{F}_x\left\{\lim_{z\to-\infty}\frac{\partial\phi}{\partial z}\right\} = \mathcal{F}_x\{0\}.$$

Bring the transform inside the limit.

$$\lim_{z\to-\infty}\mathcal{F}_x\left\{\frac{\partial\phi}{\partial z}\right\} = 0$$

Transform the partial derivative.

$$\lim_{z\to-\infty}\frac{d\Phi}{dz}=0$$

Differentiating Φ with respect to z, we obtain

$$\frac{d\Phi}{dz}(k,z,t)=A(k,t)|k|e^{|k|z}-B(k,t)|k|e^{-|k|z}.$$

In order for the boundary condition to be satisfied, we require that $B(k,t)=0$.

$$\Phi(k,z,t)=A(k,t)e^{|k|z} \tag{6}$$

Take the Fourier transform with respect to x of the boundary conditions at $z=0$ now.

$$\begin{aligned}\mathcal{F}_x\{\phi_t+g\eta\}&=\mathcal{F}_x\left\{-\frac{P}{\rho}p(x)e^{i\omega t}\right\}\\ \mathcal{F}_x\{\phi_z-\eta_t\}&=\mathcal{F}_x\{0\}\end{aligned}$$

Use the linearity property.

$$\begin{aligned}\mathcal{F}_x\{\phi_t\}+g\mathcal{F}_x\{\eta\}&=-\frac{P}{\rho}\tilde{p}(k)e^{i\omega t}\\ \mathcal{F}_x\{\phi_z\}-\mathcal{F}_x\{\eta_t\}&=0\end{aligned}$$

Transform the partial derivatives.

$$\begin{aligned}\frac{d\Phi}{dt}+gH&=-\frac{P}{\rho}\tilde{p}(k)e^{i\omega t}\\ \frac{d\Phi}{dz}-\frac{dH}{dt}&=0\end{aligned} \tag{7}$$

Differentiate both sides of the first equation with respect to t and move dH/dt to the right side.

$$\begin{aligned}\frac{d^2\Phi}{dt^2}+g\frac{dH}{dt}&=-\frac{P}{\rho}\tilde{p}(k)i\omega e^{i\omega t}\\ \frac{d\Phi}{dz}&=\frac{dH}{dt}\end{aligned}$$

Substitute the second equation into the first.

$$\frac{d^2\Phi}{dt^2}+g\frac{d\Phi}{dz}=-\frac{P}{\rho}\tilde{p}(k)i\omega e^{i\omega t} \tag{8}$$

Evaluate the derivatives of $\Phi(k,z,t)$ in equation (6).

$$\begin{aligned}\frac{d^2\Phi}{dt^2}&=A_{tt}e^{|k|z} &&\rightarrow &&\left.\frac{d^2\Phi}{dt^2}\right|_{z=0}=A_{tt}\\ \frac{d\Phi}{dz}&=A(k,t)|k|e^{|k|z} &&\rightarrow &&\left.\frac{d\Phi}{dz}\right|_{z=0}=|k|A\end{aligned}$$

Plug these expressions for the derivatives into equation (8).

$$A_{tt} + g|k|A = -\frac{P}{\rho}\tilde{p}(k) i\omega e^{i\omega t}$$

The solution to this second-order inhomogeneous ODE is

$$A(k,t) = C_1(k)\cos(t\sqrt{g|k|}) + C_2(k)\sin(t\sqrt{g|k|}) + \frac{i\omega P\tilde{p}(k)}{\rho(\omega^2 - g|k|)}e^{i\omega t}.$$

Resonance occurs if $\omega^2 = g|k|$ in which case the solution is unbounded. To determine $C_1(k)$ and $C_2(k)$, we make use of the initial conditions, $\phi(x,z,0) = 0 = \eta(x,0)$. Take the Fourier transform of both sides of these conditions.

$$\begin{aligned} \mathcal{F}_x\{\phi(x,z,0)\} = \mathcal{F}_x\{0\} \quad &\to \quad \Phi(k,z,0) = 0 \\ \mathcal{F}_x\{\eta(x,0)\} = \mathcal{F}_x\{0\} \quad &\to \quad H(k,0) = 0 \end{aligned}$$

Using the first condition, we get

$$\Phi(k,z,0) = A(k,0)e^{|k|z} = 0,$$

which implies that

$$C_1(k) + \frac{i\omega P\tilde{p}(k)}{\rho(\omega^2 - g|k|)} = 0 \quad \to \quad C_1(k) = -\frac{i\omega P\tilde{p}(k)}{\rho(\omega^2 - g|k|)}.$$

Solve equation (7) for H now.

$$H(k,t) = \frac{1}{g}\left[-\frac{P}{\rho}\tilde{p}(k)e^{i\omega t} - \frac{d\Phi}{dt}\right]\bigg|_{z=0}$$

Using the second condition, $H(k,0) = 0$, we find the second constant $C_2(k)$.

$$C_2(k) = \frac{\sqrt{g|k|}P\tilde{p}(k)}{\rho(\omega^2 - g|k|)}$$

Putting all this together and simplifying, we therefore have

$$\begin{aligned} \Phi(k,z,t) &= \frac{P\tilde{p}(k)e^{|k|z}}{\rho(\omega^2 - g|k|)}[i\omega(e^{i\omega t} - \cos(t\sqrt{g|k|}) + \sqrt{g|k|}\sin(t\sqrt{g|k|})] \\ H(k,t) &= \frac{P\tilde{p}(k)}{\rho g(g|k| - \omega^2)}[-g|k|e^{i\omega t} + g|k|\cos(t\sqrt{g|k|}) + i\omega\sqrt{g|k|}\sin(t\sqrt{g|k|})] \end{aligned}$$

where

$$\tilde{p}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} p(x)e^{-ikx}\,dx.$$

To get $\phi(x,z,t)$ and $\eta(x,t)$, take the inverse Fourier transform of Φ and H, respectively.

$$\phi(x,z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(k,z,t)e^{ikx}\,dk \quad \text{and} \quad \eta(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}H(k,t)e^{ikx}\,dk$$